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EXCEPTIONAL CASES OF INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE AND EQUATIONS OF THE FIRST KIND

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EXCEPTIONAL CASES OF INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE AND EQUATIONS OF THE FIRST KIND

F. D. Gakhov and V. I. Smagina

Integral equations with difference kernels for cases when the coefficient of the corresponding Riemann boundary value problem reverts to zero or to an infinity of an integral order are investigated. Equations of the first kind are studied, in particular. Here the Riemann-problem coefficient has a zero or a pole at the point at infinity.

In a number of studies published in recent years [2-7], the theory of convolution integral equations

$$\lambda\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_0^\infty k_1(x-t)\,\varphi(t)\,(dt)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 k_2(x-t)\,\varphi(t)\,dt=f(x), \tag{A}$$

$$-\infty<x<\infty,$$

$$\left.\begin{aligned}\lambda_1\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_1(x-t)\,\varphi(t)\,dt=f(x),\quad 0<x<\infty,\\ \lambda_2\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_2(x-t)\,\varphi(t)\,dt=f(x),\ -\infty<x<\infty,\end{aligned}\right\} \tag{B}$$

was greatly advanced. But this advancement did not affect the theory of the corresponding equations of the first kind

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} k_1 (x - t) \varphi (t) \, dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} k_2 (x - t) \varphi (t) \, dt = f (x), \quad -\infty < x < \infty, \tag{A$_0$}$$

and

$$\left. \begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_1 (x - t) \varphi (t) \, dt &= f (x), \quad 0 < x < \infty, \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2 (x - t) \varphi (t) \, dt &= f (x), \quad -\infty < x < 0, \end{aligned} \right\} \tag{B$_0$}$$

and the reason for this is not clear. It is known that equations of the convolution type are singular equations for which the Noether, rather than the Fredholm theory is valid. As Yu. I. Cherskiy [3] has shown, equations of the convolution type may be reduced to singular equations with a Cauchy kernel:

$$a (t) \varphi (t) + \frac{b (t)}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi (\tau)}{\tau - t} d\tau = f (t). \tag{0.1}$$

On the other hand it is well known that for the latter equations, by contrast to the Fredholm equation, equations of the first kind are not an independent class but are a special case of an equation of the second kind (0.1), and their solution may be obtained from solution of the latter when $a(t) \equiv 0$. The reason why this is not valid for convolution equations of the first kind and of what their uniqueness consists, has not been explained as yet, and these equations have continued to be excluded from the general theory. In the present paper we fill this gap.

Up to now the theory of convolution equations has been constructed for the so-called normal case, i.e., under the condition that the coefficient of the corresponding equation for the Riemann boundary-

value problem reverted neither to zero nor to infinity on the entire contour — the real axis. For an equation of the first kind this condition is violated at an infinitely far point of the contour. Thus, in order to include equations of the first kind in the over-all scheme of equations of the convolution type, it is necessary to construct a theory of convolution equations for exceptional cases. It was natural not to confine ourselves merely to the case when the normality is violated at a point at infinity, but to examine the problem in its entirety. This is all the more advantageous since, by contrast to an equation of the first kind, for an equation of the second kind only the finite points of the contour may prove exceptional.

Until now only one paper containing an investigation of exceptional cases has been known — V. A. Fok's paper [2] where Eq. (A,B) with a symmetric kernel was examimed.

We begin our presentation by examining the exceptional cases of the Riemann boundary-value problem when its coefficient reverts to zero or infinity of integral orders; the solution in this case is sought in the class of functions which are bounded on the contour and disappear at infinity. The corresponding theory for finite contours is well known [1, §15]. It happens that the presence of an infinite point on the contour introduces no serious changes into the theory; the need to take into account a possible singularity at infinity, however, alters the notation considerably and, viewed from the outside, the theory of exceptional cases for the real axis seems different than for a finite contour.

In solving the Riemann problem we have declined to use the auxiliary (canonical) function having zero order everywhere except for one exceptional point in the finite portion of the plane. This

last condition, making it necessary to introduce redundant factors which are cancelled in the final calculation, proves so inconvenient in the solution of specific problems that in practice it is preferred not to use the general solution formula, but to solve each problem independently by considering the form of the given coefficients. We shall introduce four polynomials with zeros concentrated in the upper or lower half-planes of each polynomial, and operate with them in such a way that the exceptional point will be at infinity. Such a form of the general solution leaves sufficient freedom for operations and permits the general solution to be used directly in the solution of specific problems. The general solution is especially convenient in those cases when the point at infinity is itself the exceptional point of the problem, which always happens in problems corresponding to integral equations of the first kind.

The obtained results are then used for the investigation of exceptional cases of convolution equations of the second kind (A) and (B), and the corresponding equations of the first kind (A_o) and (B_o). Results for equations with one kernel

$$\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_0^\infty k(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty, \qquad \text{(A,B)}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty k(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty. \qquad (A_o, B_o)$$

ensue from this as a special case.

Methods given in the papers of I. M. Rapoport [3], Yu. I. Cherskiy [4], and F. D. Gakhov and Yu. I. Cherskiy [5] are used to solve equations of the convolution type. In the present paper the theory of integral equations is essentially reduced to an investigation of the corresponding Riemann boundary-value problem. Study of the

effect of the zeros and poles of the coefficient in the Riemann problem on the number of linearly independent solutions and solvability conditions of the equations occupies a central place in this investigation. An exact quantitative characterization of this effect is obtained in the paper.

An essentially new feature is the study of the singularities of the solution at points where the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ have common zeros. These points are not singular for the coefficient

$$G(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)}$$

of the Riemann problem, but they are singular for the equations themselves. It is curious that the effect of common zeros on the solutions proves to be substantially different for equations of types (A) and (B) [cf. Section 3].

In conclusion, the problem of Noether's theorems for the exceptional cases under consideration is examined. As is known, in the corresponding cases of a singular integral equation with a Cauchy kernel (0.1) Noether's theorems are not valid. However, in his above-mentioned paper [2] V. A. Fok arrived at the conclusion that Noether's theorem on the solvability of an inhomogeneous equation (it coincides here with the corresponding Fredholm theorem) for the case considered by him remains in force. This apparent paradox is explained herein. It turns out that the reason for it is first, the exceptional choice of the data, and second, the fact that the solutions of the given and transposed equations were taken in different classes. For any other solution, except that examined by V. A. Fok, Noether's theorem on the solvability of an inhomogeneous equation will not hold. It goes without saying that the solvability condition given by Fok remains a necessary condition, but ceases to be sufficient.

1. Exceptional Cases of the Riemann Boundary-Value Problem for the Half-Plane

1.1. It is required to define functions $\Phi^+(z)$, $\Phi^-(z)$, analytic in the upper and lower half-planes, respectively, bounded in the closed half-planes, and disappearing at infinity, the limiting values of which on the contour — the real axis — satisfy the boundary condition

$$\Phi^+(x) = G(x)\Phi^-(x) + g(x). \tag{1.1}$$

Let $G(x)$ become zero of integral orders $\alpha_1, \alpha_2, \ldots, \alpha_r$, respectively, at the points $a_1, a_2, \ldots, a_r$ and become infinity of orders $\beta_1, \beta_2, \ldots, \beta_s$ at the points $b_1, b_2, \ldots, b_s$ ($\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_s$ are natural numbers).

The function $G(x)$ may then be written:

$$G(x) = \frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}} G_1(x), \quad \sum_{i=1}^{r}\alpha_i = m, \quad \sum_{j=1}^{s}\beta_j = n. \tag{1.2}$$

The points a_i will be called the zeros, and the points b_j the poles of the coefficient $G(x)$. Let us represent the function $G_1(x)$ in the form

$$G_1(x) = \frac{p_+(x)\,p_-(x)}{q_+(x)\,q_-(x)} G_2(x), \tag{1.3}$$

where $p_+(x)$, $q_+(x)$, $p_-(x)$, $q_-(x)$ are polynomials of degrees m_+, n_+, m_-, n_-, respectively, having zeros in the upper (+) and lower (-) half-planes, and G_2 is a function satisfying the Hölder condition and having zero index, with $G_2(\infty) \neq 0$. This representation is obviously possible in the case being considered.

We shall allow the same poles for the free term as for the coefficient $G(x)$:

$$g(x) = \frac{g_1(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j}}, \tag{1.4}$$

where $g_1(x)$ is a function satisfying the Holder condition at all finite points* and having at infinity the form $x^n g_2(x)$, where $g_2(x)$ is a function satisfying the Hölder condition. Subsequently, some differentiability conditions will be imposed on the functions $g_1(x)$ and $G_2(x)$ in the neighborhood of the points a_i, b_j and perhaps also at the point at infinity.

Thus we are examining a solution to the Riemann problem in the following form:

$$\Phi^+(x) = \frac{\prod_{i=1}^{r} (x - a_i)^{\alpha_i} p_+(x) p_-(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j} q_+(x) q_-(x)} G_2(x) \Phi^-(x) + \\ + \frac{q_1(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j}}, \quad -\infty < x < \infty. \tag{1.5}$$

We shall call the lowest exponent $1/z$ in the expansion of a function in the neighborhood of the point at infinity the order of the function at the point infinity. A positive order will correspond to a zero, a negative order to a pole. Accordingly, the order of $G(x)$ at infinity will be expressed by the formula

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* It is easy to see that if we allow poles for $g(x)$ different from the poles of $C(x)$, then a solution to the problem in the class of functions bounded on the contour becomes impossible.

By making use of modern papers on the solution of the Riemann problem [10,12], it would be possible without any particular difficulty to examine the boundary-value problem under the sole requirement (outside of the neighborhoods of the exceptional points) of continuity for the coefficient $G_2(x)$ and the requirement that $g_1(x)$ belong to the class L_p. But since we do not wish to complicate matters by discussions which are not related to the essence of the problem under study, we impose such restrictions as will ensure solutions that are continuous right up to the contour.

$$\nu = n + n_+ + n_- - m - m_+ - m_-.$$

The number

$$\varkappa = m_+ - n_+ \tag{1.6}$$

will be called the index of the problem.*

Let us further introduce the notation

$$h = n_- - m_-; \tag{1.7}$$

the order ν will then be expressed:

$$\nu = h - \varkappa + n - m. \tag{1.8}$$

The solution to Problem (1.5) will be sought in the class of functions bounded everywhere on the contour and disappearing at infinity.

1.2. Consider the homogeneous problem:

$$\Phi^+(x) = \frac{\prod_{i=1}^{r} (x - a_i)^{\alpha_i} p_+(x)\, p_-(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j} q_+(x)\, q_-(x)} G_2(x)\, \Phi^-(x). \tag{1.9}$$

Let us represent the function $G_2(x)$, which has an index equal to zero, in the form of a ratio

$$G_2(x) = \frac{e^{\Gamma^+(x)}}{e^{\Gamma^-(x)}}, \tag{1.10}$$

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* This concept of an index at first glance seems arbitrary and unrelated to its commonly accepted definition. This is not so however: in going from the case being considered to the normal case ($m = n = 0$, $m_+ + m_- = n_+ + n_-$), we obtain the usual definition of an index.

If, while preserving all the conditions of the problem, we carry out a conformal transformation so that the real axis converts into a closed curve (e.g., into a circle), then $\varkappa$ will be the index of the so-called reduced coefficient, i.e., of that factor which remains after the elimination of the binomials characterizing the zeros and poles of $G(x)$ on the contour [1, p. 116].

where

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln G_2(\tau) \frac{d\tau}{\tau - z}. \tag{1.11}$$

Following the general rule for the solution of the Riemann problem for exceptional cases [1, §15], we substitute (1.10) into (1.9) and write the boundary condition in the form:

$$\frac{q_-(x)}{\prod_{i=1}^{r} (x-a_i)^{\alpha_i} p_-(x)} \frac{\Phi^+(x)}{e^{\Gamma^+(x)}} = \frac{p_+(x)}{\prod_{j=1}^{s} (x-b)_j^{\beta_j} q_+(x)} \frac{\Phi^-(x)}{e^{\Gamma^-(x)}}. \tag{1.12}$$

Equation (1.12) indicates that the function

$$\frac{q_-(x)\Phi^+(x)}{\prod_{i=1}^{r} (x-a_i)^{\alpha_i} p_-(x) e^{\Gamma^+(x)}},$$

analytic in the domain D^+ and of order $m - h + 1$ at infinity, and the function

$$\frac{p_+(x)\Phi^-(x)}{\prod_{j=1}^{s} (x-b_j)^{\beta_j} q_+(x) e^{\Gamma^-(x)}},$$

analytic in the domain D^- and of order $n - \varkappa + 1$ at infinity, are an analytic continuation of each other through the contour — the real axis. Consequently they are branches of a single analytic function, which may have only one singularity in the entire plane — a pole of some order at the point at infinity. The points a_i and b_j cannot be singular points of the single analytic function, since this would contradict the assumption regarding the bounding of $\Phi^+(x)$ or of $\Phi^-(x)$.

Let us consider the two possible cases.

1. $\nu \geq 0$ ($G(x)$ has a zero of order ν at infinity). It follows from (1.8) that $n - \varkappa \geq m - h$. According to the generalized Liouville theorem

$$\frac{q_-(z)}{\prod_{i=1}^{r} (z-a_i)^{\alpha_i} p_-(z)} \frac{\Phi^+(z)}{e^{\Gamma^+(z)}} = \frac{p_+(z)}{\prod_{j=1}^{s} (z-b_j)^{\beta_j} q_+(z)} \frac{\Phi^-(z)}{e^{\Gamma^-(z)}} = P_{\varkappa-n-1}(z),$$

whence we get:

$$\Phi^+(z) = \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\, p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n-1}(z),$$

$$\Phi^-(z) = \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\, q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n-1}(z). \tag{1.13}$$

The problem has $\varkappa - n$ linearly independent solutions when $\varkappa - n >$ > 0 and a trivial zero solution when $\varkappa - n \leq 0$. As we know, the number of linearly independent solutions to the Riemann boundary-value problem in the normal (not exceptional) case in the class being considered is exactly equal to the index of the problem. The obtained formulas show that this number does not change due to the presence of zeros in $G(x)$ and decreases by the total number of poles. This result agrees with what we know for finite contours [1].

2. $\nu < 0$ ($G(x)$ has a pole of order $-\nu$ at infinity). In this case $m - h > n - \varkappa$. The general solution to the homogeneous problem (1.9) will be written in the form

$$\Phi^+(z) = \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\, p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)} P_{h-m-1}(z),$$

$$\Phi^-(z) = \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\, q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)} P_{h-m-1}(z). \tag{1.14}$$

The problem has $h - m$ linearly independent solutions when $h - m >$ > 0 and has no solutions different from the trivial one when $h - m \leq 0$.

In accordance with (1.8),

$$h - m = \varkappa - n - (-\nu).$$

Accordingly, here as well the number of solutions has been decreased by the total order of the poles (including the pole at the infinite point). Thus, with respect to the effect on the number of solutions

to the problem on the part of the zeros and poles of the coefficient, the infinite point behaves the same as the finite points.

1.3. Let us consider now the inhomogeneous problem:

$$\Phi^+(x) = \frac{\prod_{i=1}^{r} (x-a_i)^{\alpha_i} p_+(x)\, p_-(x)}{\prod_{j=1}^{s} (x-b_j)^{\beta_j} q_+(x)\, q_-(x)} G_2(x)\, \Phi^-(x) + \frac{g_1(x)}{\prod_{j=1}^{s} (x-b_j)^{\beta_j}}. \qquad (1.5)$$

1. Let $\nu \geq 0$. By virtue of the fact that the first two terms in Eq. (1.5) become zero at infinity, the minimum possible order of $g_1(x)$ at infinity is $-n+1$.

Replacing, as in the homogeneous problem, $G_2(x)$ by the ratio of two functions (1.10), we write the boundary condition in the form:

$$\underset{-n-h+1}{\frac{\prod_{j=1}^{s} (x-b_i)^{\beta_j} q_-(x)\, \Phi^+(x)}{p_-(x)\, e^{\Gamma^+(x)}}} = \underset{-m-\varkappa+1}{\frac{\prod_{i=1}^{r} (x-a_i)^{\alpha_i} p_+(x)\, \Phi^-(x)}{p_+(x)\, e^{\Gamma^-(x)}}} + \underset{-n-h+1}{\frac{g_1(x)\, q_-(x)}{p_-(x)\, e^{\Gamma^+(x)}}} \qquad (1.5')$$

(the minimum orders of the corresponding functions at infinity are written below). We isolate the main part $Q(x)$ of the expansion of the last term in the neighborhood of the point at infinity (in the case $n + h - 1 > 0$):

$$\frac{g_1(x)\, q_-(x)}{p_-(x)\, e^{\Gamma^+(x)}} = Q(x) + \psi(x),$$

where $\psi(\infty) = 0$, and $Q(x)$ is a polynomial of degree $n + h - 1$. The function $\psi(x)$ will be integrable. After replacing it by the difference between the boundary values of the analytic functions

$$\psi(x) = \Psi^+(x) - \Psi^-(x), \qquad (1.15)$$

where

$$\Psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(\tau)}{\tau - z} d\tau, \tag{1.16}$$

we reduce the boundary condition to the form:

$$\frac{\prod_{j=1}^{s} (x - b_j)^{\beta_j} q_-(x) \Phi^+(x)}{p_-(x) e^{\Gamma^+(x)}} - Q(x) - \Psi^+(x) = \frac{\prod_{i=1}^{r} (x - a_i)^{\alpha_i} p_+(x) \Phi^-(x)}{q_+(x) e^{\Gamma^-(x)}} - \Psi^-(x).$$

Applying the analytic continuation theorem and the generalized Liouville theorem and remembering that the sole singular point of the function being considered can only be at infinity, and also the fact that $-n - h \leq -m - \varkappa$ $(\nu \geq 0)$, we get:

$$\begin{aligned} \Phi^+(z) &= \frac{p_-(z) e^{\Gamma^+(z)}}{\prod_{j=1}^{s} (z - b_j)^{\beta_j} q_-(z)} [\Psi^+(z) + Q(z) + P_{\varkappa+m-1}(z)], \\ \Phi^-(z) &= \frac{q_+(z) e^{\Gamma^-(z)}}{\prod_{i=1}^{r} (z - a_i)^{\alpha_i} p_+(z)} [\Psi^-(z) + P_{\varkappa+m-1}(z)]. \end{aligned} \tag{1.17}$$

By making use of Eqs. (1.17) which yield solutions generally going to infinity at the points a_i and b_j, we construct a special solution to the inhomogeneous problem, called by L. A. Chikin [9] the canonical function of the inhomogeneous problem.

Definition. The canonical function Y(z) of the inhomogeneous problem (1.5) is a piecewise analytic function satisfying the boundary condition (1.5) having zero order everywhere in the plane (including points a_i and b_j) and possessing the lowest possible order at infinity.

In our case the canonical function Y(z) will have the form:

$$Y^+(z) = \frac{p_-(z) e^{\Gamma^+(z)}}{\prod_{j=1}^{s} (z - b_j)^{\beta_j} q_-(z)} [\Psi^+(z) + Q(z) - \widetilde{Q}_\rho(z)],$$

$$Y^{-}(z) = \frac{q_{+}(z)\, e^{\Gamma^{-}(z)}}{\prod_{i=1}^{r} (z - a_i)^{\alpha_i} p_{+}(z)} [\Psi^{-}(z) - \tilde{Q}_{\rho}(z)], \tag{1.18}$$

where $\tilde{Q}_{\rho}(z)$ is the interpolation polynomial, whose degree $\rho = m + + m - 1$.

The existence of this polynomial will be guaranteed if we require of the functions $\prod_{j=1}^{s} (x - b_j)^{\beta_j} g(x), G_1(x)$ that they have derivatives of orders α_i and β_j satisfying the Hölder condition [1, §15], at points a_i and b_j.

Since the general solution of the inhomogeneous problem is composed of some particular solution of the inhomogeneous problem and the general solution of the homogeneous problem, the general solution of Problem (1.5) is written in the form:

$$\begin{aligned} \Phi^{+}(z) &= Y^{+}(z) + \frac{\prod_{i=1}^{r} (z - a_i)^{\alpha_i} p_{-}(z)\, e^{\Gamma^{+}(z)}}{q_{-}(z)} P_{\varkappa-n-1}(z), \\ \Phi^{-}(z) &= Y^{-}(z) + \frac{\prod_{j=1}^{s} (z - b_j)^{\beta_j} q_{+}(z)\, e^{\Gamma^{-}(z)}}{p_{+}(z)} P_{\varkappa-n-1}(z). \end{aligned} \tag{1.19}$$

When $\varkappa - n > 0$, the problem has $\varkappa - n$ linearly independent solutions. In the case $\varkappa - n \leq 0$ it is necessary to set $P_{\varkappa-n-1}(z) \equiv 0$.

We note that when $\varkappa - n < 0$ the canonical function $Y(z)$ has an order at infinity of $\varkappa - n < 0$ and, consequently, ceases to be a solution to the inhomogeneous problem. However, by subjecting the free term to conditions it is possible to achieve an increase in the order at infinity of the function $Y(z)$ by $n - \varkappa$ units and thereby again make the canonical function a solution to the inhomogeneous problem. In

order for these conditions to be fulfilled, we must require that the functions $x^k g_1(x)$ and $G_2(x)$ have at infinity derivatives of orders up to $n - \varkappa$ satisfying the Holder condition.

2. Now let $\nu < 0$. The lowest possible order at infinity of the function $g_1(x)$ will be $n - \varkappa - m + 1$. Consequently, in the boundary condition (1.5') the function

$$\frac{g_1(x)\, q_-(x)}{p_-(x)\, e^{\Gamma^+(x)}}$$

will have an order at infinity equal to $-m - \varkappa + 1$. In this case after separating out the main part of the expansion of

$$\frac{g_1(x)\, q_-(x)}{p_-(x)\, e^{\Gamma^+(x)}}$$

in the neighborhood of the point at infinity when $m + \varkappa - 1 > 0$ the boundary condition is written in the form:

$$\frac{\prod_{j=1}^{s}(x - b_j)^{\beta_j}\, q_-(x)\, \Phi^+(x)}{p_-(x)\, e^{\Gamma^+(x)}} - \Psi^+(x) = \frac{\prod_{i=1}^{r}(x - a_i)^{\alpha_i}\, p_+(x)\, \Phi^-(x)}{q_+(x)\, e^{\Gamma^-(x)}} - \Psi^-(x) + Q(x).$$

The canonical function of the inhomogeneous problem is expressed in terms of the interpolation polynomial in the following way:

$$\begin{aligned} Y_1^+(z) &= \frac{p_-(z)\, e^{\Gamma^+(z)}}{\prod_{j=1}^{s}(z - b_j)^{\beta_j}\, q_-(z)}\, [\Psi^+(z) - \widetilde{Q}_\rho(z)], \\ Y_1^-(z) &= \frac{q_+(z)\, e^{\Gamma^-(z)}}{\prod_{i=1}^{r}(z - a_i)^{\alpha_i}\, p_+(z)}\, [\Psi^-(z) - Q(z) - \widetilde{Q}_\rho(z)], \end{aligned} \tag{1.20}$$

while the general solution to Problem (1.5) takes on the form:

$$\begin{aligned} \Phi^+(z) &= Y_1^+(z) + \frac{\prod_{i=1}^{r}(z - a_i)^{\alpha_i}\, p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)}\, P_{h-m-1}(z), \\ \Phi^-(z) &= Y_1^-(z) + \frac{\prod_{j=1}^{s}(z - b_j)^{\beta_j}\, q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)}\, P_{h-m-1}(z). \end{aligned}$$

When h - m > 0 the problem has h - m linearly independent solutions. When $h - m \leq 0$ the polynomial $P_{h-m-1}(z)$ should be set indentically equal to zero, and when $h - m < 0$ the free term g(x) must be required to meet m - h conditions of the same form as in the preceding case. When these conditions are fulfilled the inhomogeneous problem (1.5) will have a unique solution.

2. Integral Equations with Two Kernels

2.1. Let us examine an integral equation of the convolution type of the second kind:

$$\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_0^\infty k_1(x-t)\varphi(t)dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 k_2(x-t)\varphi(t)\,dt = f(x), \tag{A}$$

where λ is the piecewise constant:

$$\lambda = \begin{cases}\lambda_1, & x>0,\\ \lambda_2, & x<0.\end{cases}$$

After denoting, as usual, functions identically equal to zero for positive (negative) x with the aid of the subscripts - (+), we can give to Eq. (A) the following form:

$$\begin{gathered}\lambda_1\varphi_+(x) - \lambda_2\varphi_-(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_1(x-t)\varphi_+(t)\,dt - \\ -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_2(x-t)\varphi_-(t)\,dt = f(x), \quad \varphi(x) = \varphi_+(x) - \varphi_-(x).\end{gathered} \tag{2.1}$$

We effect a Fourier transformation of this equation by making use of the following well-known theorem: in order for some function $\Phi(x)$, given on the real axis and satisfying the conditions for which the formulas of Fourier transforms are valid, to be a boundary value of a function analytic in the upper (lower) half-plane with a uniformly bounded integral $\int_{-\infty}^\infty |\Phi(x+iy)|^2\,dx$, it is necessary and sufficient

that its inverse Fourier transform $\varphi(x)$ be equal to zero for $x < 0$ ($x > 0$). We then obtain:

$$[\lambda_1 + K_1(x)]\,\Phi^+(x) - [\lambda_2 + K_2(x)]\,\Phi^-(x) = F(x), \tag{2.2}$$

or

$$\Phi^+(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)}\,\Phi^-(x) + \frac{F(x)}{\lambda_1 + K_1(x)}. \tag{2.3}$$

Thus, the solution of an integral equation of class (A) is equivalent to constructing functions $\Phi^+(z)$ and $\Phi^-(z)$, analytic for $y > 0$, respectively, in accordance with Boundary Condition (2.3).

We shall assume the kernels $k_1(x)$ and $k_2(x)$ and the free term $f(x)$ of Eq. (A) to be such that the conditions imposed upon the coefficients of Problem (2.3) in Section 1 are fulfilled. We will seek the solution $\varphi(x)$ in the class of functions whose Fourier transforms belong in that class of functions in which the Riemann problem was solved in Section 1, i.e., are function satisfying the Hölder condition.

According to (2.1), the solution to the integral equation is determined form the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\Phi^+(t) - \Phi^-(t)]\,e^{-ixt}\,dt. \tag{2.4}$$

We shall assume that the functions $\lambda_1 + K_1(x)$, $\lambda_2 + K_2(x)$ may have zeros and that these zeros may be at different, as well as at coincident points of the contour. Let us write the expansion of these functions, after distinguishing the coincident zeros:

$$\left.\begin{aligned} \lambda_1 + K_1(x) &= \prod_{j=1}^{s}(x - b_j)^{\beta_j}\prod_{k=1}^{l}(x - c_k)^{\gamma_k}K_{11}(x), \\ \lambda_2 + K_2(x) &= \prod_{i=1}^{r}(x - a_i)^{\alpha_i}\prod_{k=1}^{l}(x - c_k)^{\gamma_k}K_{12}(x), \end{aligned}\quad \sum_{k=1}^{l}\gamma_k = l.\right\} \tag{2.5}$$

It is possible that individual points $c_k (k = 1, 2, \ldots, t)$ may coincide either with a_i or with b_j. This will correspond to the case when the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ have a common zero of different multiplicity. We do not distinguish points a_i and b_j coincident with some of the points c_k because such a coincidence has no effect on the solvability conditions and the number of solutions to the problem.

From Eq. (2.2) and the condition of the finiteness of the solution on the contour, there ensues the fact that for the solvability of the problem and, consequently, of Eq. (A) as well, it is necessary that the function F(x) revert to a zero of order γ_k at all points c_k, i.e., F(x) must have the form:

$$F(x) = \prod_{k=1}^{t} (x - c_k)^{\gamma_k} F_1(x).$$

This requires the fulfillment $\gamma_1 + \gamma_2 + \ldots + \gamma_t = l$ of the conditions

$$F^{(j_k)}(c_k) = 0 \quad (j_k = 0, 1, \ldots, \gamma_k - 1). \tag{2.6}$$

According to the relationship

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ixt} dt,$$

they will be the conditions for the right side f(x) of Eq. (A).

The coefficient G(x) of Riemann Problem (2.3) is represented in the form

$$\frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)} = \frac{\prod_{i=1}^{r} (x - a_i)^{\alpha_i} p_+(x) p_-(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j} q_+(x) q_-(x)} G_2(x).$$

Since the functions $K_1(x)$ and $K_2(x)$ vanish at infinity, the coefficient G(x) is bounded there; accordingly, the point at infinity is not a singular point of G(x).

Let us assume that Conditions (2.6) are fulfilled. Then Boundary-Value Problem (2.3) will have the form:

$$\Phi^{+}(x)=\frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}p_{+}(x)\,p_{-}(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}q_{+}(x)\,q_{-}(x)}G_2(x)\,\Phi^{-}(x)+\frac{g_1(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}}. \tag{2.7}$$

This problem was solved in Section 1 and its general solution written in form (1.9):

$$\begin{aligned}\Phi^{+}(z)&=Y^{+}(z)+\frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}p_{-}(z)\,e^{\Gamma^{+}(z)}}{q_{-}(z)}P_{\varkappa-n-1}(z),\\ \Phi^{-}(z)&=Y^{-}(z)+\frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}q_{+}(z)\,e^{\Gamma^{-}(z)}}{p_{+}(z)}P_{\varkappa-n-1}(z).\end{aligned} \tag{2.8}$$

From here, using Formula (2.4), we obtain the solution of the original integral equation. It is easy to verify that $\Phi^{\pm}(z)$ satisfy all the imposed conditions ($\Phi^{\pm}(x)$ satisfy the Hölder condition and the integral $\int|\Phi(x+iy)|^2dx$ is uniformly bounded).

Thus, for the solvability of Eq. (A) it is necessary that the Fourier transform of the free term of the equation satisfy Conditions (2.6). If these conditions are fulfilled, then, as indicated by Formulas (2.8), when $\varkappa - n \leq 0$ the polynomial $P_{\varkappa-n-1}(z)$ must be set identically equal to zero, in which case it is necessary to require the free term to meet $n - \varkappa$ more conditions in the case $\varkappa - n < 0$. When these are fulfilled, the integral equation will have a unique solution.

Consequently, in the case of the solvability of Eq. (A) the number of linearly independent solutions of Eq. (A) is equal to the difference between the index of the Riemann problem and the total number of poles of its coefficient, and is independent of the zeros of the coefficient as well as of the common zeros of the functions $\lambda_1 + K_1(x)$,

and $\lambda_2 + K_2(x)$. The latter increase the number of solvability conditions.

2.2. Integral Equations of the First Kind with Two Kernels.

Let us examine the integral equation of the first kind

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty k_1(x-t)\varphi(t)\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 k_2(x-t)\varphi(t)\,dt = f(x), \qquad (A_0)$$
$$-\infty < x < \infty.$$

It may be obtained from Eq. (A) when $\lambda \equiv 0$. In this case, the Fourier transform of Eq. (A_0) leads to the solution of the boundary-value problem:

$$\Phi^+(x) = \frac{K_2(x)}{K_1(x)}\Phi^-(x) + \frac{F(x)}{K_1(x)}, \quad -\infty < x < \infty. \qquad (2.9)$$

Coefficient $G(x)$ of Problem (2.9) is the ratio of functions which vanish at infinity and consequently, in contrast to the preceding case, may have a zero or a pole of some order at the point at infinity.

Let

$$K_1(x) = \frac{\Psi_1(x)}{x^\lambda}, \quad K_2(x) = \frac{\Psi_2(x)}{x^\mu},$$

where the functions $\Psi_1(x)$ and $\Psi_2(x)$ have zero order at infinity. Two cases may arise, depending on whether the difference $\nu = \mu - \lambda$ is negative or positive. For generality we shall assume that exceptional points are also present at infinite distance. Let the functions $K_1(x)$ and $K_2(x)$ have the representations:

$$K_1(x) = \prod_{j=1}^{s}(x-b_j)^{\beta_j}\prod_{k=1}^{t}(x-c_k)^{\gamma_k}K_{11}(x),$$
$$K_2(x) = \prod_{i=1}^{r}(x-a_i)^{\alpha_i}\prod_{k=1}^{t}(x-c_k)^{\gamma_k}K_{12}(x).$$

In addition to the common zeros at points c_k of multiplicity γ_k, the functions $K_1(x)$ and $K_2(x)$ have at infinity a common zero of order equal to $\min(\lambda,\mu)$.

The Riemann-problem coefficient will have the form:

$$G(x)=\frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}p_+(x)p_-(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}q_+(x)q_-(x)}G_2(x).$$

It follows from (2.9) that this problem and integral equation (A_0) will be solvable in the required class of functions, if it is required that F(x) fulfill Conditions (2.6) at points c_k which are common zeros of the functions $K_1(x)$ and $K_2(x)$. In the case of an equation of the first kind there are added to these conditions d = min (λ,μ) + 1 more conditions, imposed on the behavior at infinity, since the functions $K_1(x)$ and $K_2(x)$ have at infinity a common zero of order min (λ,μ). Accordingly, function F(x) should satisfy Conditions (2.6) and have an order no lower than d at infinity. If these conditions are fulfilled, the boundary-value problem (2.9) assumes the form:

$$\Phi^+(x)=\frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}p_+(x)p_-(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}q_+(x)q_-(x)}G_2(x)\Phi^-(x)+\frac{g_1(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}}. \tag{2.10}$$

The solution of this problem was given in Section 1. When $\nu \geq 0(\mu \geq \lambda)$ it is written in the form:

$$\begin{aligned}\Phi^+(z)&=Y^+(z)+\frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}p_-(z)e^{\Gamma^+(z)}}{q_-(z)}P_{\varkappa-n-1}(z),\\ \Phi^-(z)&=Y^-(z)+\frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}q_+(z)e^{\Gamma^-(z)}}{p_+(z)}P_{\varkappa-n-1}(z),\end{aligned} \tag{2.11}$$

and when $\nu < 0(\mu < \lambda)$, in the form:

$$\begin{aligned}\Phi^+(z)&=Y_1^+(z)+\frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}p_-(z)e^{\Gamma^+(z)}}{q_-(z)}P_{h-m-1}(z),\\ \Phi^-(z)&=Y_1^-(z)+\frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}q_+(z)e^{\Gamma^-(z)}}{p_+(z)}P_{h-m-1}(z).\end{aligned} \tag{2.12}$$

The solution to the original integral equation in both cases may be obtained by substituting Expressions (2.11) and (2.12) into Formula (2.4). Regarding the number of solutions it is possible to derive the same result as for an equation of the second kind. The number of solvability conditions increases by comparison with the solvability conditions for an equation of the second kind by $\underline{d}$.

2.3. Equation with One Kernel. Let us give the solution to an integral equation that is frequently encountered in applications

$$\lambda\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_0^\infty k(x-t)\varphi(t)\,dt=f(x),\quad 0<x<\infty, \tag{A,B}$$

by considering it as a special case of Eq. (A)* ($k_1(x) \equiv k(x)$, $k_2(x) \equiv \equiv 0$; $\lambda_1 = \lambda$; $\lambda_2 = 1$).

Fourier transformation yields:

$$\Phi^+(x)=\frac{\Phi^-(x)}{\lambda+K(x)}+\frac{F(x)}{\lambda+K(x)}. \tag{2.13}$$

The coefficient $G(x) = 1/\lambda + K(x)$ of the obtained boundary-value problem has neither a zero nor a pole ($\nu = 0$) at infinity. We shall consider the case when kernel $k(x)$ of the integral equation is such that the function $\lambda + K(x)$ may become zero at the contour;** then the coefficient $G(x)$ may be presented in the form

$$G(x)=\frac{p_+(x)\,p_-(x)}{\prod_{j=1}^{n}(x-b_j)^{\beta_j}q_+(x)\,q_-(x)}G_2(x).$$

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* It would also be possible to consider it as a special case of paired equations (B).

** Since here $k_2(x) \equiv 0$, the function $\lambda_2 + K_2(x)$ has no zeros, and, consequently, the complications associated with the possibility of common zeros of the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ disappear.

Thus solving the integral equation (A,B) is equivalent to solving a Riemann boundary-value problem of the following form:

$$\Phi^{+}(x)=\frac{p_{+}(x)\,p_{-}(x)}{\prod\limits_{j=1}^{s}(x-b_j)^{\beta_j}q_{+}(x)\,q_{-}(x)}G_2(x)\,\Phi^{-}(x)+\frac{g_1(x)}{\prod\limits_{j=1}^{s}(x-b_j)^{\beta_j}}. \qquad (2.14)$$

We obtain the general solution to the homogeneous problem (2.14) as a special case of the solution of Problem (1.5) for $\nu = 0$, when there are no zeros for the coefficient $G(x)$ at the points a_i, which as we know do not affect the number of linearly independent solutions of the problem. We have:

$$\Phi^{+}(z)=\frac{p_{-}(z)\,e^{\Gamma^{+}(z)}}{q_{-}(z)}P_{\varkappa-n-1}(z),$$
$$\Phi^{-}(z)=\frac{\prod\limits_{j=1}^{s}(z-b_j)^{\beta_j}q_{+}(z)\,e^{\Gamma^{-}(z)}}{p_{+}(z)}P_{\varkappa-n-1}(z). \qquad (2.15)$$

In order to solve the inhomogeneous problem we shall construct the canonical function.

The interpolation polynomial $\tilde{Q}(z)$ should in this case satisfy the conditions:

$$\tilde{Q}^{(i)}(b_j)=\Psi^{+(i)}(b_j), \quad j=1,2,\ldots,s;\ i=0,1,\ldots,\beta_j-1.$$

The degree of the polynomial $\rho = n - 1$.

The canonical function of the inhomogeneous problem is expressed in terms of the interpolation polynomial in the following way:

$$Y^{+}(z)=\frac{p_{-}(z)\,e^{\Gamma^{+}(z)}}{\prod\limits_{j=1}^{s}(z-b_j)^{\beta_j}q_{-}(z)}[\Psi^{+}(z)+Q(z)-\tilde{Q}_{\rho}(z)],$$
$$Y^{-}(z)=\frac{q_{+}(z)\,e^{\Gamma^{-}(z)}}{p_{+}(z)}[\Psi^{-}(z)-\tilde{Q}_{\rho}(z)], \qquad (2.16)$$

while the general solution will have the form:

$$\Phi^+(z) = Y^+(z) + \frac{p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n-1}(z),$$

$$\Phi^-(z) = Y^-(z) + \frac{\prod_{j=1}^{s} (z-b_j)^{\beta_j} q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n-1}(z).$$

When $\varkappa - n > 0$ the problem has $\varkappa - n$ linearly independent solutions. In the case $\varkappa - n < 0$ Eq. (A,B) is solvable only when $n - \varkappa$ conditions are fulfilled.

The solution to integral equation (A,B) is found using the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi^+(t)\, e^{-ixt} dt. \tag{2.17}$$

When $\lambda \equiv 0$ we obtain an equation of the first kind:

$$\frac{1}{\sqrt{2\pi}} \int_0^{\infty} k(x-t)\, \varphi(t)\, dt = f(x), \quad 0 < x < \infty, \tag{A_0,B_0}$$

to which it is possible to give the form:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x-t)\, \varphi_+(t)\, dt - f(x) = \varphi_-(x), \quad -\infty < x < \infty.$$

Fourier transformation of the last equation leads to a boundary-value problem of the type:

$$\Phi^+(x) = \frac{\Phi^-(x)}{K(x)} + \frac{F(x)}{K(x)}. \tag{2.18}$$

As usual, we represent the boundary-value problem (2.18) in the form

$$\Phi^+(x) = \frac{p_+(x)\, p_-(x)}{\prod_{j=1}^{s} (x-b_j)^{\beta_j} q_+(x)\, q_-(x)} G_2(x)\, \Phi^-(x) + \frac{g_1(x)}{\prod_{j=1}^{s} (x-b_j)^{\beta_j}}.$$

Since K(x) is a function which vanishes at infinity, the coefficient $G(x) = \frac{1}{K(x)}$ of the obtained problem, by contrast to the corresponding coefficient for an equation of the second kind, always has a pole of some order at infinity.

Let the order of K(x) at infinity be λ. The order of G(x) at infinity is equal to $\nu = -\lambda < 0$. We obtain the general solution to

this problem as a special case of the solution to Problem (1.5) in the case $\nu < 0$, when there are no zeros for the coefficient G(x) at the points a_i. It is written in the form

$$\Phi^+(z) = Y^+(z) + \frac{p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n+\nu-1}(z),$$

$$\Phi^-(z) = Y^-(z) + \frac{\prod_{j=1}^{s} (z-b_j)^{\beta_j} q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n+\nu-1}(z).$$

We find the solution to Integral Equation (A_0, B_0) using Formula (2.17).

The number of linearly independent solutions to Integral Equation (A_0, B_0) in the case $\varkappa - n + \nu > 0$ is decreased by $-\nu$ in comparison with the corresponding equation of the second kind ($-\nu$ is the order of the pole at infinity of the function G(x)).

<u>Example 1</u>. Let us examine the equation of the first kind

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(x-t)\varphi(t)\,dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 k_2(x-t)\varphi(t)\,dt = f(x), \tag{2.19}$$

where

$$k_1(x) = \begin{cases} 0, & x > 0, \\ \sqrt{2\pi}\,(e^{3x} - e^{2x}), & x < 0, \end{cases}$$

$$k_2(x) = \begin{cases} -\sqrt{2\pi}\, i e^{-2x}, & x > 0, \\ 0, & x < 0, \end{cases}$$

$$f(x) = \begin{cases} 0, & x > 0, \\ \sqrt{2\pi}\,(e^{3x} - e^{2x}), & x < 0. \end{cases}$$

Fourier transformation of (2.19) yields:

$$K_1(x) = \frac{1}{(x-2i)(x-3i)}, \quad K_2(x) = \frac{1}{x+2i}, \quad F(x) = \frac{1}{(x-2i)(x-3i)}.$$

Boundary-Value Problem (2.9) will be written:

$$\Phi^+(x) = \frac{(x-2i)(x-3i)}{x+2i}\Phi^-(x) + 1.$$

The coefficient G(x) has a pole of the first order ($\nu = -1$) at the point at infinity,

$$m_+ = 2, \quad n_+ = 0, \quad \varkappa = m_+ - n_+ = 2, \quad d = 2 \;\; (\min(\lambda, \mu) = 1).$$

$F(x)$ has a second-order zero at infinity; consequently, the necessary condition for the solvability of the equation is fulfilled.

The homogeneous problem

$$\Phi^{+}(x) = \frac{(x-2i)(x-3i)}{x+2i}\Phi^{-}(x)$$

has the following solution in the class of functions disappearing at infinity:

$$\Phi^{+}(z) = \frac{C}{z+2i},$$
$$\Phi^{-}(z) = \frac{C}{(z-2i)(z-3i)}.$$

The number of linearly independent solutions to Problem (2.19) is one less than the index at infinity since $G(x)$ has a pole of first order.

The inhomogeneous problem in the class of functions disappearing at infinity has the following solution

$$\Phi^{+}(z) = \frac{C}{z+2i},$$
$$\Phi^{-}(z) = \frac{C-z-2i}{(z-2i)(z-3i)}.$$

The solution to the integral equation is found according to the formula:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\Phi^{+}(t)-\Phi^{-}(t)]\,e^{-ixt}dt,$$

$$\varphi(x) = \begin{cases} -\sqrt{2\pi}\,iCe^{-2x}, & x>0, \\ \sqrt{2\pi}\,C(e^{2x}-e^{3x}) - 4i\sqrt{2\pi}\,e^{2x} + 5i\sqrt{2\pi}\,e^{3x}, & x<0. \end{cases}$$

The equation proves to be solvable for the right-hand side selected. For example, if we take

$$f(x) = \begin{cases} \sqrt{2\pi}\,(5ie^{3x}-4ie^{2x}), & x<0 \\ 0, & x>0, \end{cases} \tag{2.20}$$

then

$$F(x) = \frac{x+2i}{(x-2i)(x-3i)}.$$

The necessary condition for the solvability of Eq. (2.19) is not fulfilled; consequently, the problem has no solutions which vanish at infinity. The solution to the problem

$$\Phi^{+}(x)=\frac{(x-2i)(x-3i)}{x+2i}\Phi^{-}(x)+x+2i$$

in the class of functions bounded at infinity will be:

$$\Phi^{+}(z)=\frac{-z+C}{z+2i},$$

$$\Phi^{-}(z)=\frac{-z-(z+2i)^2+C}{(z-2i)(z-3i)}.$$

For no choice of the constant C does the solution vanish at infinity; consequently, the equation with the right side given in Eq. (2.20) has no solution integrable on the real axis.

3. Paired Integral Equations

3.1. Let us examine integral equations of the form

$$\begin{aligned}&\lambda_1\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_1(x-t)\varphi(t)\,dt=f(x), \qquad 0<x<\infty,\\&\lambda_2\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_2(x-t)\varphi(t)\,dt=f(x), \qquad -\infty<x<0.\end{aligned} \tag{B}$$

By introducing the auxiliary functions $\omega_+(x)$, $\omega_-(x)$ [5 and 6], we shall complete the definition of Eq. (B) so that both of them will be given on the entire axis. We will have:

$$\begin{aligned}&\lambda_1\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_1(x-t)\varphi(t)\,dt-f(x)=\omega_-(x),\\&\lambda_2\varphi(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_2(x-t)\varphi(t)\,dt-f(x)=\omega_+(x),\end{aligned} \qquad -\infty<x<\infty. \tag{3.1}$$

Effecting Fourier transformation, we get:

$$\begin{aligned}&[\lambda_1+K_1(x)]\,\Phi(x)-F(x)=\Omega^-(x),\\&[\lambda_2+K_2(x)]\,\Phi(x)-F(x)=\Omega^+(x),\end{aligned} \qquad -\infty<x<\infty. \tag{3.2}$$

From this it follows that

$$\Phi(x) = \frac{\Omega^+(x) + F(x)}{\lambda_2 + K_2(x)} = \frac{\Omega^-(x) + F(x)}{\lambda_1 + K_1(x)}. \tag{3.3}$$

Consequently, the limiting values of the auxiliary piecewise analytic function $\Omega(z)$ should satisfy the following boundary condition:

$$\Omega^+(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)} \Omega^-(x) + \frac{\lambda_2 - \lambda_1 + K_2(x) - K_1(x)}{\lambda_1 + K_1(x)}, \quad -\infty < x < \infty. \tag{3.4}$$

Thus, solving the paired integral equations is equivalent to constructing functions $\Omega^+(z)$, $\Omega^-(z)$, analytic when $y > 0$ and $y < 0$, respectively, according to Boundary Condition (3.4).

The solution to original Integral Equation (B) is determined according to the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\Omega^+(t) + F(t)}{\lambda_2 + K_2(t)} e^{-ixt} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\Omega^-(t) + F(t)}{\lambda_1 + K_1(t)} e^{-ixt} dt. \tag{3.5}$$

As in the case of Integral Equation (A), we shall assume that the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ can become zero at individual points of the contour. Let us write out the expansion of these functions after separating out their common zeros c_k:

$$\begin{aligned} \lambda_1 + K_1(x) &= \prod_{j=1}^{s} (x - b_j)^{\beta_j} \prod_{k=1}^{t} (x - c_k)^{\gamma_k} K_{11}(x), \\ \lambda_2 + K_2(x) &= \prod_{i=1}^{r} (x - a_i)^{\alpha_i} \prod_{k=1}^{t} (x - c_k)^{\gamma_k} K_{12}(x). \end{aligned} \tag{3.6}$$

As before, the coefficient $G(x)$ may be presented in the form

$$\frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)} = \frac{\prod_{i=1}^{r} (x - a_i)^{\alpha_i} p_+(x) p_-(x)}{\prod_{j=1}^{s} (x - b_j)^{\beta_j} q_+(x) q_-(x)} G_2(x). \tag{3.7}$$

At the point at infinity $G(x)$ has neither zeros nor poles $(m + \varkappa = n + h$.

Let us ascertain in what class of functions the solutions to Problem (3.4) should be sought in order that Formulas (3.5) determine the solution of Integral Equation (B) in the required class.

Since the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ revert to zero on the contour, in order for Integral (3.5) to have meaning it is necessary to require that the functions $\Omega^+(z)$ and $\Omega^-(z)$ when solving the homogeneous problem ($F(x) \equiv 0$) and the functions $\Omega^+(x) + F(x)$ and $\Omega^-(x) + F(x)$ when solving the inhomogeneous problem, which functions occur in the numerator in (3.5), revert to a zero of the same multiplicity and at the same points as the denominators of these fractions. From the equality

$$\frac{\Omega^+(x) + F(x)}{\lambda_2 + K_2(x)} = \frac{\Omega^-(x) + F(x)}{\lambda_1 + K_1(x)} \tag{3.3'}$$

it follows that these conditions are automatically fulfilled for those zeros a_i and b_j which are not common to the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$. Consequently, there remains to require only the fulfillment of the conditions

$$\begin{gathered} \Omega^{\pm}(\infty) = 0, \\ \Omega^{+(s)}(c_k) = 0, \\ \Omega^{-(s)}(c_k) = 0 \\ (s = 0, 1, \ldots, \gamma_k - 1, \quad k = 1, 2, \ldots, l) \end{gathered} \tag{3.8}$$

for the solutions of the homogeneous problem and of the conditions

$$\begin{gathered} \frac{d^s}{dx^s}[\Omega^{\pm}(x) + F(x)]_{x=c_k} = 0, \\ \Omega^{\pm}(\infty) = 0 \end{gathered} \tag{3.9}$$

for the inhomogeneous problem.

In this way it is sufficient to require fulfillment of Conditions (3.8) and (3.9) for only one of the functions $\Omega^+(x)$ or $\Omega^-(x)$, since it will be automatically fulfilled for the other by virtue of (3.3').

Let us proceed to the solution of Problem (3.4).

Homogeneous Problem:

$$\Omega^+(x) = \frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}\, p_+(x)\, p_-(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}\, q_+(x)\, q_-(x)}\, G_2(x)\, \Omega^-(x). \tag{3.10}$$

A problem of this form was solved in Section 1. Its general solution is given by the formula

$$\begin{aligned} \Omega^+(z) &= \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\, p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)}\, P_{\varkappa-n-1}(z), \\ \Omega^-(z) &= \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\, q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)}\, P_{\varkappa-n-1}(z). \end{aligned} \tag{3.11}$$

We shall require fulfillment of Conditions (3.8). For this the polynomial $P_{\varkappa-n-1}(z)$ must obviously be taken in the form

$$\prod_{k=1}^{l}(z-c_k)^{\gamma_k}\, P_{\varkappa-n-l-1}(z),$$

i.e., the solution to Problem (3.10) in the required class of functions is written in the form:

$$\begin{aligned} \Omega^+(z) &= \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i} \prod_{k=1}^{l}(z-c_k)^{\gamma_k}\, p_-(z)}{q_-(z)}\, P_{\varkappa-n-l-1}(z), \\ \Omega^-(z) &= \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j} \prod_{k=1}^{l}(z-c_k)^{\gamma_k}\, q_+(z)}{p_+(z)}\, P_{\varkappa-n-l-1}(z). \end{aligned} \tag{3.12}$$

In this way the number of linearly independent solutions to Problem (3.10) is decreased not just by the number of poles of the coefficient G(x) on the contour, but also by the number of common zeros (counting also their multiplicity) of the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$.

Formulas (3.12) give the solution to Problem (3.10) when $\varkappa - n - l > 0$ and, consequently, in this case the homogeneous integral equation (B) has $\varkappa - n - l$ linearly independent solutions. In the case $\varkappa - n - l \leq 0$, however, the homogeneous problem (3.10) and the

homogeneous integral equation (B) have no solutions differing from the identically zero solution.

Inhomogeneous problem. In solving the inhomogeneous problem we shall proceed from the fact that its general solution is composed of some particular solution of the inhomogeneous problem and the general solution of the homogeneous problem.

As in Section 1, we shall construct the canonical function of the inhomogeneous problem. But in the case being considered the canonical function must also satisfy Conditions (3.9). In order to satisfy these conditions, the interpolation polynomial $Q_\rho(z)$ must be taken with degree ρ equal to $m + n + l - 1$.

The canonical function will then be expressed in terms of the interpolation polynomial in the following way:

$$
\begin{aligned}
Y^+(z) &= \frac{p_-(z)e^{\Gamma^+(z)}}{\prod_{j=1}^{s}(z-b_j)^{\beta_j} q_-(z)}\left[\Psi^+(z) + Q(z) - \widetilde{Q}_\rho(z)\right], \\
Y^-(z) &= \frac{q_+(z)e^{\Gamma^-(z)}}{\prod_{i=1}^{r}(z-a_i)^{\alpha_i} p_+(z)}\left[\Psi^-(z) - \widetilde{Q}_\rho(z)\right].
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\Omega^+(z) &= Y^+(z) + \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\prod_{k=1}^{t}(z-c_k)^{\gamma_k} p_-(z)e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n-l-1}(z), \\
\Omega^-(z) &= Y^-(z) + \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\prod_{k=1}^{t}(z-c_k)^{\gamma_k} q_+(z)e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n-l-1}(z).
\end{aligned}
\tag{3.14}
$$

In this way, when $\varkappa - n - l > 0$ the solution to the inhomogeneous problem (3.4) and also to Integral Equation (B) will be linearly dependent upon $\varkappa - n - l$ arbitrary constants. When $\varkappa - n\ l \leq 0$, the polynomial $P_{\varkappa-n-l-1}(z)$ must be set identically equal to zero. When $\varkappa - n - l < 0$ the canonical function $Y(z)$ has order $\varkappa - n - l < 0$ at infinity and, consequently, ceases to be a solution to the inhomogeneous

problem. However, by subjecting the free term $F(x)$ to $n + l - \varkappa$ conditions [since $F(x)$ is the Fourier transform of $f(x)$, the conditions imposed upon $F(x)$ will be at the same time conditions imposed upon the free term $f(x)$ of Integral Equation (B)], it is possible to secure a higher order at infinity for the function $Y(z)$ by $n + l - \varkappa$ units and again make the canonical function $Y(z)$ a solution to the inhomogeneous problem. This means that in the case $\varkappa - n - l < 0$ the inhomogeneous problem (3.4) and, consequently, Integral Equation (B) are solvable in a unique way only when $n + l - \varkappa$ solvability conditions have been fulfilled.

We note that the effect of the common zeros of the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ on the solvability conditions and the number of solutions is considerably different for equations of types (A) and (B). For Eq. (A), as ensues from the study in Section 2, the presence of common zeros imposes conditions upon the free term of the equation which must necessarily be fulfilled in order for the equation to be able to have a solution. It is not possible to satisfy these conditions by choosing from among the arbitrary constants available in the general solution. Thus, regardless of the value of the index of the equation, the equation will be solvable only when the right side is specially chosen. The common zeros will also result in solvability conditions for Eq. (B), but they may be satisfied by an appropriate choice of the arbitrary constants entering into the general solution. Consequently, for a sufficiently large index ($\varkappa - n - l \geq 0$) the equation will undoubtedly be solvable.

3.2. Paired integral equations of the first kind. Let us consider the solution to paired integral equations of the first kind obtained from Eq. (B) when $\lambda_1 = \lambda_2 = 0$:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(x-t)\varphi(t)\,dt = f(x), \quad -\infty < x < 0. \tag{B_0}$$

The boundary-value problem for the type of equations under consideration is obtained from (3.4) after setting $\lambda_1 = \lambda_2 = 0$:

$$\Omega^+(x) = \frac{K_2(x)}{K_1(x)}\Omega^-(x) + \frac{K_2(x) - K_1(x)}{K_1(x)}F(x), \quad -\infty < x < \infty. \tag{3.15}$$

In contrast to the Riemann boundary-value problem (3.4) obtained for the corresponding integral equation of the second kind where the coefficient

$$G(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)}$$

of the boundary-value problem is bounded at infinity, in the given case this coefficient is the ratio of functions disappearing at infinity, and consequently may have a zero or a pole of some order at the point at infinity. Let us assume

$$K_1(x) = \frac{\Psi_1(x)}{x^\lambda}, \quad K_2(x) = \frac{\Psi_2(x)}{x^\mu},$$

where the functions $\Psi_1(x)$ and $\Psi_2(x)$ have zero order at infinity. Two different cases may arise, depending upon whether the difference $\nu = \mu - \lambda$ is negative or positive.

Let

$$K_1(x) = \prod_{j=1}^{s}(x-b_j)^{\beta_j}\prod_{k=1}^{t}(x-c_k)^{\gamma_k}K_{11}(x),$$

$$K_2(x) = \prod_{i=1}^{r}(x-a_i)^{\alpha_i}\prod_{k=1}^{t}(x-c_k)^{\gamma_k}K_{12}(x).$$

Then

$$\frac{K_2(x)}{K_1(x)} = \frac{\prod_{i=1}^{r}(x-a_i)^{\alpha_i}\, p_+(x)\, p_-(x)}{\prod_{j=1}^{s}(x-b_j)^{\beta_j}\, q_+(x)\, q_-(x)}G_2(x);$$

and the solution to Integral Equation (B_0) is determined according to Formula (3.5) when $\lambda_1 = \lambda_2 = 0$:

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\Omega^{+}(t)+F(t)}{K_2(t)}e^{-ixt}\,dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\Omega^{-}(t)+F(t)}{K_1(t)}e^{-ixt}\,dt. \tag{3.16}$$

1. $\nu > 0 (\mu > \lambda)$. In this case the coefficient $G(x)$ of the boundary-value problem (3.15) has a zero of order ν at infinity. But the functions $K_1(x)$ and $K_2(x)$ which are now in the denominator of the fractions

$$\frac{\Omega^{-}(x)+F(x)}{K_1(x)}$$

and

$$\frac{\Omega^{+}(x)+F(x)}{K_2(x)},$$

in addition to common zeros of the same multiplicity at the points c_k, have a common zero of order λ at infinity. Therefore, in determining the class in which the solution to (3.15) must be sought, in addition to the restrictions that were imposed upon the solution in the case of the analogous equation of the second kind, it must be required that the function $\Omega^{+}(z)$ have a zero of order $\lambda + 1$ at infinity. The function $\Omega^{-}(z)$, on the other hand, will have the required order $\mu + 1$ at infinity by virtue of Eq. (3.3) when $\lambda_1 = \lambda_2 = 0$.

In this way, the general solution to the homogeneous problem (3.15) is written in the form:

$$\begin{aligned}\Omega^{+}(z)&=\frac{\prod\limits_{i=1}^{r}(z-a_i)^{\alpha_i}\prod\limits_{k=1}^{l}(z-c_k)^{\gamma_k}p_{-}(z)\,e^{\Gamma^{+}(z)}}{q_{-}(z)}P_{\varkappa-n-l-\lambda-1}(z),\\ \Omega^{-}(z)&=\frac{\prod\limits_{j=1}^{s}(z-b_j)^{\beta_j}\prod\limits_{k=1}^{l}(z-c_k)^{\gamma_k}q_{+}(z)\,e^{\Gamma^{-}(z)}}{p_{+}(z)}P_{\varkappa-n-l-\lambda-1}(z).\end{aligned} \tag{3.17}$$

The number of linearly independent solutions to the homogeneous problem for an equation of the first kind is decreased, just as in the case of an equation of the second kind, by the number of poles of the coefficient $G(x)$ on the real axis, and also by the number of common

zeros of the functions $K_1(x)$ and $K_2(x)$.

This means that in the case $\varkappa - n + \lambda < 0$, the homogeneous problem (3.15), and together with it the homogeneous integral equation (B_0), have no non-trivial solutions.

In solving the inhomogeneous problem we shall argue in the same way as in the preceding case, but now it is necessary to require of the cononical function $Y(z)$ of the nonhomogeneous problem, in addition to fulfilling Conditions (3.9), that if have a zero of order λ at infinity. Therefore the polynomial $\tilde{Q}_\rho(z)$ should be taken with degree $\rho = m + n + l + \lambda - 1$.

The general solution of the inhomogeneous problem (3.15) will be expressed in terms of the canonical function of the inhomogeneous problem in the following way:

$$\begin{aligned} \Omega^+(z) &= Y^+(z) + \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\prod_{k=1}^{t}(z-c_k)^{\gamma_k}p_-(z)e^{\Gamma^+(z)}}{q_-(z)}P_{\varkappa-n-l-\lambda-1}(z), \\ \Omega^-(z) &= Y^-(z) + \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\prod_{k=1}^{t}(z-c_k)\,q_+(z)e^{\Gamma^-(z)}}{p_+(z)}P_{\varkappa-n-l-\lambda-1}(z). \end{aligned} \tag{3.18}$$

Formulas (3.18) show that when $\varkappa - n - l - \lambda > 0$, Problem (3.15) and Integral Equation (B_0) have $\varkappa - n - l - \lambda$ linearly independent solutions. In the case $\varkappa - n - l - \lambda < 0$ Problem (3.15) and Integral Equation (B_0) will have a unique solution only when $n + l + \lambda - \varkappa$ solvability conditions are fulfilled.

2. $\nu < 0$ $(\mu < \lambda)$. In this case the coefficient $G(x)$ of the boundary-value problem (3.15) has a pole of order ν at infinity, while the functions $K_1(x)$ and $K_2(x)$ have there a common zero of order μ. Consequently, the number of linearly independent solutions of the homogeneous problem (3.15) should decrease by the total number of poles of the coefficient $G(x)$ on the contour, i.e., by $n + |\nu|$, and

also by the number of common zeros of the functions $K_1(x)$ and $K_2(x)$, i.e., by $l + \mu$. In this way, when $\nu < 0$ the number of linearly independent solutions to the homogeneous problem (3.15) will be

$$\varkappa - n - l + \nu - \mu = \varkappa - n - l - \lambda.$$

By arguing in the same way as before, we come to the conclusion that the canonical function of the inhomogeneous problem should in this case have a zero of order μ at infinity, while the interpolation polynomial should be taken with degree

$$\rho = m + n + l - \nu + \mu - 1 = m + n + l + \lambda - 1.$$

The general solutions to the inhomogeneous problem (3.15) will have the form:

$$\begin{aligned} \Omega^+(z) &= Y^+(z) + \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i}\prod_{k=1}^{t}(z-c_k)^{\gamma_k}\, p_-(z)\, e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n-l-\lambda-1}(z), \\ \Omega^-(z) &= Y^-(z) + \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j}\prod_{k=1}^{t}(z-c_k)^{\gamma_k}\, q_+(z)\, e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n-l-\lambda-1}(z). \end{aligned} \tag{3.19}$$

The conclusions regarding the number of linearly independent solutions when $\varkappa - n - l - \lambda > 0$ and solvability conditions when $\varkappa - n - l - \lambda < 0$ remain the same as for the case $\nu > 0$.

Example 2. Let us examine the integral equation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(x-t)\,\varphi(t)\,dt = f(x), \qquad 0 < x < \infty,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(x-t)\,\varphi(t)\,dt = f(x), \quad -\infty < x < 0,$$

where

$$k_1(x) = \begin{cases} 0, & x > 0, \\ \sqrt{2\pi}\,(e^{3x} - e^{2x}), & x < 0, \end{cases} \qquad k_2(x) = \begin{cases} -i\sqrt{2\pi}\,e^{-2x}, & x > 0, \\ 0, & x < 0, \end{cases}$$

$$f(x) = \begin{cases} -\frac{\sqrt{2\pi}}{4}\,e^{-2x}, & x > 0, \\ \frac{\sqrt{2\pi}}{4}\,e^{2x}, & x < 0. \end{cases}$$

Fourier transformation yields:

$$K_1(x) = \frac{1}{(x-2i)(x-3i)}, \quad K_2(x) = \frac{1}{x+2i}, \quad F(x) = \frac{1}{x^2+4}.$$

The boundary value problem (3.15) which corresponds to the equation in question is written in the form:

$$\Omega^{+}(x)=\frac{(x-2i)(x-3i)}{x+2i}\Omega^{-}(x)+\frac{x-3i}{(x+2i)^{2}}-\frac{1}{x^{2}+4}. \tag{3.20}$$

The coefficient $G(x)$ has a pole of first order at infinity ($\nu = -1$). The functions $K_1(x)$ and $K_2(x)$ have a common first-order zero at the point at infinity,

$$m_{+} = 2, \quad n_{+} = 0, \quad \varkappa = m_{+} - n_{+} = 2.$$

Representing the boundary condition in the form

$$(x+2i)\Omega^{+}(x)-\frac{x-3i}{x+2i}=(x-2i)(x-3i)\Omega^{-}(x)-\frac{1}{x-2i}$$

and making use of analytic continuation and the Liouville theorem, we find that the general solution to (3.20) in the class of functions disappearing at infinity will have the form:

$$\begin{aligned}\Omega^{+}(z)&=\frac{1}{z+2i}\left[\frac{z-3i}{z+2i}+C\right],\\ \Omega^{-}(z)&=\frac{1}{(z-2i)(z-3i)}\left[\frac{1}{z-2i}+C\right].\end{aligned} \tag{3.21}$$

The solution to the integral equation is found according to the formula:

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\Omega^{+}(t)+F(t)}{K_{2}(t)}e^{-ixt}\,dt.$$

Since the function $K_2(x)$ has a first-order zero at infinity, the function $\Omega^{+}(x) + F(x)$ must have at the point at infinity a zero of at least second order. From this condition we find that $C = -1$.

But when $C = -1$ the solution of (3.21) assumes the form:

$$\Omega^{+}(z)=\frac{-5i}{z^{2}+4},$$

$$\Omega^{-}(z)=\frac{1-z+2i}{(z-2i)^{2}(z-3i)},$$

and we obtain:

$$\varphi(x)=\begin{cases}5\sqrt{2\pi}\,e^{-2x}, & x>0,\\ i\sqrt{2\pi}\,e^{2x}, & x<0.\end{cases}$$

Thus the solvability condition resulting from the presence of a common zero of the functions $K_1(x)$ and $K_2(x)$ can be satisfied by a

choice of the arbitrary constant entering into the general solution and the given integral equation proves to be unconditionally and unambiguously solvable (cf. Example 1 in Section 2).

4. The Noether Theorems

The three Noether theorems* [4, p. 55] are, in the normal case, valid for singular integral equations, which also include, along with equations having a Cauchy kernel, equations of the convolution type.

Theorem 1 regarding the finiteness of the number of solutions also remains valid for exceptional cases. Let us see what the case is with the other two theorems.

We take Eq. (A) of the second kind as the initial equation. We shall use Yu. I. Cherskiy's method [3] to form the transposed equation.

Let us write Eq. (A) in the equivalent form:

$$\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_1(x-t)+k_2(x-t)}{2}\varphi(t)\,dt + \\ + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_2(x-t)-k_2(x-t)}{2}\varphi(t)\operatorname{sgn} t\,dt = f(x). \tag{A}$$

The homogeneous transposed equation will have the form:

$$\lambda\tilde{\varphi}(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_1(t-x)+k_2(t-x)}{2}\tilde{\varphi}(t)\,dt + \\ + \frac{\operatorname{sgn} x}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k_1(x-t)-k_2(t-x)}{2}\tilde{\varphi}(t)\,dt = 0, \tag{A'}$$

---

* These theorems are valid for complete equations in which also there is a regular operator contained as the addend. For the further negative result which is being established, examination of the more simple characteristic equations is sufficient.

or

$$\begin{aligned}
&\lambda_1\tilde{\varphi}(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_1(t-x)\tilde{\varphi}(t)\,dt=0, \qquad 0<x<\infty,\\
&\lambda_2\tilde{\varphi}(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k_2(t-x)\tilde{\varphi}(t)\,dt=0, \quad -\infty<x<0.
\end{aligned} \tag{A'}$$

Thus, a paired integral equation of type (B) will be an equation conjugate for (A). After noting that the variables x and t in the kernels of the integrals have changed their positions by comparison with the normal convolution expression,* we find that the solution of the transposed equation (A') will be equivalent to the solution of the following boundary-value problem:

$$\tilde{\Omega}^+(x)=\frac{\lambda_1+K_1(-x)}{\lambda_2+K_2(-x)}\tilde{\Omega}^-(x). \tag{4.1}$$

Preserving all the notations of Section 2, we assume that the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$ can be presented in the form of (2.5); then (4.1) will have the form:

$$\tilde{\Omega}^+(x)=\frac{\prod_{i=1}^{r}(x+a_i)^{\alpha_i}p_+(-x)\,p_-(-x)}{\prod_{j=1}^{s}(x+b_j)^{\beta_j}q_+(-x)\,q_-(-x)}G_2(-x)\tilde{\Omega}^-(x). \tag{4.2}$$

The zeros and poles of the coefficient $G(x)$ of the Riemann problem corresponding to the transposed equation have changed, but their number and multiplicity remain as before. The "+" and "-" type polynomials which enter into the coefficient of the boundary condition (4.2) have exchanged roles, so that $p_+(-x)$ and $q_+(-x)$ will have zeros in the lower half-plane, while $p_-(-x)$ and $q_-(-x)$ will have zeros in the upper. Consequently, the index of the transposed equation (A') will, by

---

* A substantial misprint was allowed in Eqs. (A') of M. G. Kreyn's article [8, p. 117]: the opposite sign was indicated for the arguments of the kernels.

definition, be equal to

$$\varkappa' = m_- - n_-.$$

Since G(x) has neither zeros nor poles at the point at infinity ($\nu = 0$),

$$m + m_+ + m_- = n + n_+ + n_-.$$

From this the following relationship between the indices of the transposed equations is obtained:

$$\varkappa' = -\varkappa - (m - n).$$

As follows from the investigations in Section 2, the inhomogeneous equation (A) is solvable in the case $\varkappa - n \geq 0$ when l conditions are fulfilled, while when $\varkappa - n < 0$, the fulfillment of $n - \varkappa + l$ conditions is required for its solvability. These conditions may be presented in terms of the right side of an equation of f(x) in the following form

$$\int_{-\infty}^{\infty} f(x)\,\psi_j(x)\,dx = 0. \tag{4.3}$$

Should Noether's second theorem prove to be valid, then $\psi_j(x)$ would have to be a complete system of linearly independent solutions to the transposed equation. By comparing the number of solvability conditions and the number of linearly independent solutions to the transposed equation, it is easy to show that this is impossible.

For example, let $\varkappa - n < 0$; then, as we have seen, the number of solvability conditions (4.3) of Eq. (A) equals $n - \varkappa + l$. But the number of solutions to the transposed equation will be equal to

$$\varkappa' - n - l = -\varkappa - m - l. \tag{4.4}$$

The difference between these numbers $m + n + 2l$ is always positive for exceptional cases ($\min(m, n, l) > 0$), and therefore the set of functions $\psi_j(x)$ entering into the solvability conditions (4.3) is not

exhausted by the complete system of solutions to the transposed equation (A'). Accordingly, Noether's second theorem is always invalid for exceptional cases.

As we proceed to the third theorem regarding the difference between the number of solutions to the given and transposed homogeneous equations, let us note first of all that these equations cannot be simultaneously solvable for the exceptional cases being considered. In fact, the initial homogeneous equation (A) is solvable only for positive index $n > n$, while the equation conjugated to it (A') is solvable only for negative index $\varkappa < -(m + l)$ when $\varkappa$ lies in the range $-(m + l) < \varkappa < n$, both equations are simultaneously insolvable. Thus, the difference in the numbers of solutions to these equations coincides with the number of solutions of one of these equations and therefore cannot be equal to the index of the equation. Accordingly, Noether's third theorem also does not hold for exceptional cases.

Note that the conclusions that have been reached are valid under the condition that the solutions to the given and transposed equations are sought in one and the same class of functions on the real axis. If functions having apolar singularity on the contour are admitted as a solution to the Riemann problem and, consequently, nonintegrable functions (for kernels satisfying the conditions for which the corresponding integrals exist) are admitted as a solution to the integral equation, then the question will require new study.

The question of the solvability of the integral equation

$$\varphi(x) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k(|x - t|)\, \varphi(t)\, dt = f(x), \quad 0 < x < \infty, \tag{4.5}$$

with a symmetrical kernel for the exceptional case when the function $1 + K(x)$ has $2p$ real roots and, consequently, the coefficient

$G(x) = \frac{1}{1 + K(x)}$ of the corresponding boundary-value problem has 2p real poles has been examined by V. A. Fok [2]. He proved that only when p conditions of the form

$$\int_0^\infty f(x)\,\gamma_r(x)\,dx = 0, \quad r = 0, 1, \ldots, p-1,$$

are fulfilled does Eq. (4.5) have a unique solution which remains bounded and tends to zero at infinity. In this case, the functions $\gamma_r(x)$ are p linearly independent solutions to the homogeneous equation in the class of functions not integrable on the real axis. Accordingly, here Noether's second theorem proves to be valid. The contradiction with the conclusions reached above regarding the absolute unfulfillability of this theorem for exceptional cases is explained by the fact that here the solution to the initial equation and its transposed equation (for the case of the symmetrical kernel in question they coincide) are taken in different classes. In addition, there occurs here a special choice of the data. In our notations the Riemann problem coefficient may be written:

$$\frac{1}{1+K(x)} = \frac{(x-ib)^p\,(x+ib)^p}{\prod\limits_{k=1}^{p}(x-x_k)(x+x_k)}\,G_2(x).$$

Here

$$p_+(x) = (x-ib)^p, \quad p_-(x) = (x+ib)^p, \quad q_+(x) = q_-(x) \equiv 1,$$

$$m = 0, \quad n = 2p, \quad m_+ = p, \quad m_- = p, \quad n_+ = n_- = 0,$$

$$\varkappa = m_+ - n_+ = p.$$

For the solvability of the inhomogeneous Riemann problem in the class of functions bounded on the contour and disappearing at infinity, $n - \varkappa = p$ conditions must be fulfilled. At the same time when poles are admitted at the points x_k, $-x_k$ for $\Phi^+(x)$ the homogeneous problem has p linearly independent solutions.* Homogeneous Equation (4.5) has p solutions which cannot be integrated on the axis.

* The number p will be the index of the problem in the class of solutions being considered [1, p. 453, Eq. (45.9)].

Thus the legality of Noether's second theorem for the given case is explained, except for the fact that the solutions to the transposed equations are taken in different classes, also by the special condition

$$m = l = 0, \quad \varkappa = \frac{n}{2} = p.$$

A similar argument could have been made if the paired equation (E) has been taken as the initial equation.

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NSA	6
DIA	9
AID	2
OTS	2
AEC	2
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AFCRL (CRXLR)	1

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RADC (RAY)	1
AFWL (WLF)	1
AFMTC (MTW)	1
ASD (ASYIM)	2